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A condition for a closed one-form to be exact

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Abstract. A condition for a closed one-form to be exact, the one-form having values in Euclidean space, on a compact surface without boundary, is given in the case where the surface has suitable differentiable automorphisms. Tori and hyperelliptic curves, with holomorphic automorphisms, are in this case. A local representation formula for surfaces in Euclidean space is then globalized. A condition for a local surface of constant mean curvature to be global, can be written using a harmonic Gauss map.

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1. Introduction

The theory of de Rham cohomology is related to the theory of surfaces in Euclidean space. A surface is a smooth map f , from a smooth orientable connected two-dimensional manifold M without boundary, to an n -dimensional Euclidean space \mathbb{R}^n . The differential df is an exact one-form on M with values in \mathbb{R}^n . Hence, a surface represents a boundary of the first de Rham cohomology group of one-forms on M , with values in \mathbb{R}^n . The f is reconstructed from its differential by the integral formula

$$f(p) = \int_{\gamma} df + f(p_0),$$

with a curve γ starting at $p_0 \in M$ and ending at $p \in M$.

If M is simply connected, then a one-form is exact if and only if the one-form is closed. In this case, there are several ways to construct differentials of surfaces. For example, the Weierstrass-Enneper representation formula for minimal surfaces in \mathbb{R}^3 constructs a differential of a minimal surface in \mathbb{R}^3 ,

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from a meromorphic function and a holomorphic one-form on a Riemann surface (see Osserman [13]). The Kenmotsu formula constructs a differential of a surface in \mathbb{R}^3 , from a non-zero mean curvature function and a Gauss map (see Kenmotsu [8]). These researches have been taken over by many researchers and developed in a variety of methods (see, for example, Hoffman and Osserman [6], Konopelchenko [12], Taimanov [18], [17], [16], [15], Friedrich [4], Pedit and Pinkall [14], and Burstall, Ferus, Leschke, Pedit and Pinkall [2]). Except [6], these researches are related with the spinorial or twistorial formulation of the theory of surfaces (see, for example, Bryant [1], Friedrich [5], Kamberov, Norman, Pedit, and Pinkall [7]).

In the case where the topology of M is more complicated, these formulae do not construct an exact one-form in general, but a closed one-form. This motivates us to study a condition for a closed one-form to be exact.

We assume that M is compact without boundary and of genus g . We embed \mathbb{R}^n into the Clifford algebra $C\ell_n$. We denote the first de Rham cohomology group of $C\ell_n$ -valued one-forms on M by Rh^1 , and the cohomology class where a closed one-form η belongs by $[\eta]$. We define a pairing $(\ , \) : \text{Rh}^1 \times \text{Rh}^1 \rightarrow C\ell_n$ by

$$([\eta], [\xi]) := \int_M \eta \wedge \xi.$$

The dimension of Rh^1 is $2g$. Let $\delta_1, \dots, \delta_{2g}$ be a basis of Rh^1 . It is well-known that a closed one-form ω on M is exact, if and only if

$$([\omega], \delta_i) = 0$$

for all $\delta_1, \dots, \delta_{2g} \in \text{Rh}^1$. In general, it is difficult to find $\delta_1, \dots, \delta_{2g}$ and calculate the above $2g$ integrals.

To ease this difficulty, we assume that M has suitable differentiable automorphisms. We denote by $H_1(M, \mathbb{Z})$ the first homology group of M with integer coefficients. For a closed curve γ in M , we denote by $[\gamma]$ the homology class where γ belongs. Let $a_1, \dots, a_g, b_1, \dots, b_g$ be closed curves in M such that $[\![a_1]\!], \dots, [\![a_g]\!], [\![b_1]\!], \dots, [\![b_g]\!]$ is a canonical basis of $H_1(M, \mathbb{Z})$. We denote by E_g the g by g unit matrix. Let J_{2g} be the $2g$ by $2g$ matrix defined by

$$J_{2g} := \begin{pmatrix} O & E_g \\ -E_g & O \end{pmatrix}.$$

We denote by $\text{Sp}(g, \mathbb{Z})$ the symplectic group of $2g$ by $2g$ matrices with entries in \mathbb{Z} . For a matrix N , we denote its transpose by N^T . Then $\text{Sp}(g, \mathbb{Z}) = \{X \mid X J_{2g} X^T = J_{2g}\}$. Let \mathcal{A} be the group of differentiable automorphisms of M . A representation $h = (h_{jk}) : \mathcal{A} \rightarrow \text{Sp}(g, \mathbb{Z})$ is defined by the equation

$$\begin{aligned} &([\mu(a_1)] \quad \dots \quad [\mu(a_g)] \quad [\mu(b_1)] \quad \dots \quad [\mu(b_g)]) \\ &= ([a_1] \quad \dots \quad [a_g] \quad [b_1] \quad \dots \quad [b_g]) h(\mu). \end{aligned}$$

We decompose the matrix $J_{2g} h^T - h J_{2g}$ into a diagonal matrix $B(\mu) = (b_{ij}(\mu))$, and a matrix $C(\mu) = (c_{ij}(\mu))$ such that the entries of the main diagonal are zero. Then, $J_{2g} h^T(\mu) - h(\mu) J_{2g} = B(\mu) + C(\mu)$. Let $\tilde{C}(\mu) =$

$(\tilde{c}_{ij}(\mu)) = (|c_{ij}(\mu)|)$ and $\Phi: \text{Cl}_n \rightarrow \mathbb{R}^1$ be the projection. Then we have the following condition.

Theorem 1.1. *We assume that there exist $\mu_1, \dots, \mu_m \in \mathcal{A}$ and $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ is positive definite,
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ is negative definite.

Let ω be a one-form on M with

$$\sum_{l=1}^m r_l (\omega \wedge \mu_l^* \omega - \mu_l^* \omega \wedge \omega) \neq 0.$$

A one-form ω with values in \mathbb{R}^n is exact, if and only if ω is closed and

$$\Phi \left(\sum_{l=1}^m r_l ([[\omega]], [\mu_l^* \omega]) - ([[\mu_l^* \omega]], [[\omega]]) \right) = 0.$$

If $\sum_{l=1}^m r_l (\omega \wedge \mu_l^* \omega - \mu_l^* \omega \wedge \omega)$ is a non-zero exact one-form, then we understand that ω is an exact one-form, without integration. We see examples in the case where M is a square torus (Corollary 4.1), a hexagonal torus (Corollary 4.2), and a hyperelliptic curve with affine plane model

$$\left\{ (z, w) \in \mathbb{C} \times \mathbb{C} \mid w^2 = z^{2(g+1)} - 1 \right\}$$

(Corollary 5.1).

We identify \mathbb{R}^4 with quaternions \mathbb{H} . For $a \in \mathbb{H}$, we denote its conjugate by \bar{a} . Then we have the following similar condition.

Theorem 1.2. *We assume that there exist $\mu_1, \dots, \mu_m \in \mathcal{A}$ and $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ is positive definite,
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ is negative definite.

Let ω be a one-form on M with

$$\sum_{l=1}^m r_l (\omega \wedge \mu_l^* \bar{\omega} - \mu_l^* \omega \wedge \bar{\omega}) \neq 0.$$

A one-form ω with values in \mathbb{H} is exact, if and only if ω is closed and

$$\sum_{l=1}^m r_l ([[\omega]], [\mu_l^* \bar{\omega}]) - ([[\mu_l^* \omega]], [[\bar{\omega}]]) = 0.$$

By Theorem 1.2, we have a property of a period of a one-form. We denote the inner product of \mathbb{R}^4 by $\langle \cdot, \cdot \rangle$. Let η be a one-form with values in $\text{Im } \mathbb{H}$, such that

$$\int_{a_1} \eta \neq 0, \quad \int_{a_2} \eta = \dots = \int_{a_g} \eta = \int_{b_1} \eta = \dots = \int_{b_g} \eta = 0.$$

We can consider η as a differential of a singly-periodic surface in \mathbb{R}^3 .

Corollary 1.3. *We assume that there exist $\mu \in \mathcal{A}$ and $r \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. $rB(\mu) - |r|\tilde{C}(\mu)$ is positive definite,
2. $rB(\mu) + |r|\tilde{C}(\mu)$ is negative definite.

Let η be an $\text{Im } \mathbb{H}$ -valued one-form such that

$$\int_{a_1} \eta \neq 0, \quad \int_{a_2} \eta = \cdots = \int_{a_g} \eta = \int_{b_1} \eta = \cdots = \int_{b_g} \eta = 0.$$

We assume that

$$r(\eta \wedge \mu^* \bar{\eta} - \mu^* \eta \wedge \bar{\eta}) \neq 0.$$

Then

$$\left\langle \int_{a_1} \eta, \int_{b_1} \mu^* \eta \right\rangle \neq 0.$$

Returning the initial motivation, we combine Theorem 1.2 and the construction of a closed one-form for a surface in \mathbb{R}^4 in [2]. We denote the complex structure of M by J . For a one-form ω on M , we define $*\omega := \omega \circ J$. We denote the set of real parts of quaternions by $\text{Re } \mathbb{H}$ and the set of imaginary parts of quaternions by $\text{Im } \mathbb{H}$. If f is an immersion, then there exists a map $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$ such that $*df = Ndf$. We call a non-constant map $f: M \rightarrow \mathbb{R}^4$ a *surface* if there exists $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$ such that $*df = Ndf$. Then we have the following condition.

Corollary 1.4. *We assume that there exist $\mu_1, \dots, \mu_m \in \mathcal{A}$ and $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ is positive definite,
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ is negative definite.

We assume that maps $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$ and $\mathcal{H}: M \rightarrow \mathbb{H} \setminus \{0\}$, and a non-zero one-form ω on M satisfy

$$-2\omega \bar{\mathcal{H}} = *dN + N dN, \quad 2\omega \wedge d\bar{\mathcal{H}} = d* dN + dN \wedge dN,$$

$$\sum_{l=1}^m r_l (\omega \wedge \mu_l^* \bar{\omega} - \mu_l^* \omega \wedge \bar{\omega}) \neq 0,$$

$$\sum_{l=1}^m r_l ([[\omega]], [\mu_l^* \bar{\omega}]) - ([[\mu_l^* \omega]], [[\bar{\omega}]]) = 0.$$

Then there exists a surface $f: M \rightarrow \mathbb{R}^4$ with mean curvature vector field \mathcal{H} , such that $df = \omega$ and $*df = Ndf$.

If f takes values in $\text{Im } \mathbb{H} \cong \mathbb{R}^3$, then $*df = Ndf = -df N$. Applying Corollary 1.2 to surfaces of constant mean curvature, we have the following condition.

Corollary 1.5. *We assume that there exist $\mu_1, \dots, \mu_m \in \mathcal{A}$ and $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ is positive definite,
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ is negative definite.

We assume that a non-conformal harmonic map $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$

$$\sum_{l=1}^m r_l (N * dN \wedge \mu_l^*(N * dN) - \mu_l^*(N * dN) \wedge (N * dN)) \neq 0,$$

$$\sum_{l=1}^m r_l ([\llbracket N * dN \rrbracket, \llbracket \mu_l^*(N * dN) \rrbracket] - (\llbracket \mu_l^*(N * dN) \rrbracket, \llbracket N * dN \rrbracket]) = 0.$$

Then there exists a surface $f: M \rightarrow \text{Im } \mathbb{H} \cong \mathbb{R}^3$ of non-zero constant mean curvature, such that $*df = N df = -df N$.

2. Clifford algebra-valued one-forms

Throughout this paper, we assume that all manifolds, maps, and differential forms are smooth. In this section, we show an analog of a relation between the periods of two complex-valued, closed one-forms (see Farkas and Kra [3], III.2.3. Proposition). Then we prove Theorem 1.1.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of \mathbb{R}^n . The Clifford algebra Cl_n is the algebra generated by e_1, \dots, e_n subject to the relation

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

We denote the projection $\text{Cl}_n \rightarrow \mathbb{R}^1$ by Φ . Let $\langle u, v \rangle$ be the inner product of u and $v \in \mathbb{R}^n$ and $|u| := \langle u, u \rangle^{1/2}$ the norm of \mathbb{R}^n . Let $u := \sum_{i=1}^n u_i e_i$, $v := \sum_{i=1}^n v_i e_i \in \mathbb{R}^n \subset \text{Cl}_n$ ($u_1, \dots, u_n, v_1, \dots, v_n \in \mathbb{R}$). Then

$$uv = -\langle u, v \rangle + \sum_{i < j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} e_i e_j$$

We identify the Clifford algebra and the exterior algebra in a natural manner. Then $uv = -\langle u, v \rangle + u \wedge v$. Hence u and v are linearly independent over \mathbb{R} , if and only if $uv - vu \neq 0$. Let $\alpha: \text{Cl}_n \rightarrow \text{Cl}_n$ be the automorphism which extends $\alpha(u) = -u$ for $u \in \mathbb{R}^n$. The map $\delta: \text{Cl}_2 \rightarrow \mathbb{H}$, which extends

$$\delta(1) = 1, \quad \delta(e_1) = i, \quad \delta(e_2) = j, \quad \delta(e_1 e_2) = k,$$

is an isomorphism between Cl_2 and \mathbb{H} .

Let M be a compact oriented two-dimensional manifold without boundary. We assume that the genus of M is g . For closed curves γ_1 and γ_2 in M , we denote $\gamma_1 \cdot \gamma_2$ the intersection number of γ_1 and γ_2 . Let $\pi_1(M) = \pi_1(M, p_0)$ be the fundamental group of M with base point $p_0 \in M$. For a closed curve u with initial and end point p_0 in M , we denote the inverse curve of u by u^{-1} , and the homotopy class that u represents by $[u]$. The map from $\pi_1(M)$ to $H_1(M, \mathbb{Z})$ defined by $[u] \mapsto \llbracket u \rrbracket$ is a surjective group homomorphism. We fix simple closed curves

$$\{a_1, \dots, a_g, b_1, \dots, b_g\}$$

in M with initial and end point $p_0 \in M$, such that all curves are disjoint from each other, except p_0 , that

$$a_i \cdot a_j = b_i \cdot b_j = 0, \quad a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij} \quad (i, j = 1, \dots, g),$$

and that

$$[a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}] = 1.$$

The ordered cycles

$$[a_1], \dots, [a_g], [b_1], \dots, [b_g]$$

form a canonical basis of $H_1(M, \mathbb{Z})$. We assume that $[\gamma] \in H_1(M, \mathbb{Z})$ and $[\eta] \in \text{Rh}^1$. Then we can define a map $V_{[\gamma]}: \text{Rh}^1 \rightarrow \text{Cl}_n$ by

$$V_{[\gamma]}([\eta]) := \int_{\gamma} \eta.$$

We set

$$\begin{aligned} P(\eta) &= (P_1(\eta) \quad \cdots \quad P_g(\eta) \quad P_{g+1}(\eta) \quad \cdots \quad P_{2g}(\eta)) \\ &:= (V_{[a_1]}([\eta]) \quad \cdots \quad V_{[a_g]}([\eta]) \quad V_{[b_1]}([\eta]) \quad \cdots \quad V_{[b_g]}([\eta])). \end{aligned}$$

Lemma 2.1. *Let η and ξ be closed one-forms on M with values in Cl_n . Then*

$$([\eta], [\xi]) = P(\eta) J_{2g} P(\xi)^T \quad (2.1)$$

Proof. Let $\psi: U \rightarrow M$ be the universal covering. Then there exists a simply connected set \tilde{M} with boundary $\partial \tilde{M}$ in U such that

$$\partial \tilde{M} = \tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1} \tilde{b}_1^{-1} \cdots \tilde{a}_g \tilde{b}_g \tilde{a}_g^{-1} \tilde{b}_g^{-1},$$

$$\psi(\tilde{a}_i) = a_i, \quad \psi(\tilde{a}_i^{-1}) = a_i^{-1}, \quad \psi(\tilde{b}_i) = b_i, \quad \psi(\tilde{b}_i^{-1}) = b_i^{-1} \quad (i = 1, \dots, g).$$

(see Figure 1).

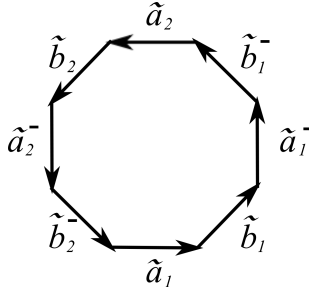


FIGURE 1. \tilde{M} in the case where $g = 2$.

We fix a point $z_0 \in \tilde{M}$ such that $\psi(z_0) = p_0$. We denote a curve with initial point z_0 and end point z in \tilde{M} by γ . We can define a map $f: \tilde{M} \rightarrow \text{Cl}_n$ by

$$f(z) := \int_{\gamma} \psi^* \eta.$$

By Stokes' theorem, we have

$$\begin{aligned} ([\eta], [\xi]) &= \int_M \eta \wedge \xi = \int_{\tilde{M}} \psi^*(\eta \wedge \xi) = \int_{\tilde{M}} df \wedge \psi^*\xi = \int_{\partial\tilde{M}} f\psi^*\xi \\ &= \sum_{m=1}^g \int_{a_m+b_m+a_m^-+b_m^-} f\psi^*\xi. \end{aligned} \quad (2.2)$$

For a point $z \in \tilde{a}_m$, we define $z' \in \tilde{a}_m^-$ by $\psi(z) = \psi(z')$. Let $\tilde{\gamma}$ be a curve in \tilde{M} with initial point z_0 and end point z' . Then

$$\begin{aligned} \int_{\tilde{a}_m+\tilde{a}_m^-} f\psi^*\xi &= \int_{\tilde{a}_m} \left(\int_{\gamma} \psi^*\eta - \int_{\tilde{\gamma}} \psi^*\eta \right) \psi^*\xi \\ &= - \int_{\tilde{b}_m} \psi^*\eta \int_{\tilde{a}_m} \psi^*\xi = - \int_{b_m} \eta \int_{a_m} \xi = -P_{g+m}(\eta)P_m(\xi). \end{aligned}$$

Similarly, for a point $z \in \tilde{b}_m$, we define $z' \in \tilde{b}_m^-$ by $\psi(z) = \psi(z')$. Then

$$\begin{aligned} \int_{\tilde{b}_m+\tilde{b}_m^-} f\psi^*\xi &= \int_{\tilde{b}_m} \left(\int_{\gamma} \psi^*\eta - \int_{\tilde{\gamma}} \psi^*\eta \right) \psi^*\xi \\ &= \int_{\tilde{a}_m} \psi^*\eta \int_{\tilde{b}_m} \psi^*\xi = \int_{a_m} \eta \int_{b_m} \xi = P_m(\eta)P_{g+m}(\xi). \end{aligned}$$

By (2.2), we have (2.1). □

Proof of Theorem 1.1. For a closed curve γ in M , we have

$$V_{[\gamma]}([\tau^*\omega]) = \int_{\gamma} \tau^*\omega = \int_{\tau(\gamma)} \omega = V_{[\tau(\gamma)]}([\omega]).$$

Hence

$$P(\mu_l^*\omega) = P(\omega)h(\mu_l).$$

Then

$$([\omega], [\mu_l^*\omega]) = P(\omega)J_{2g}P(\mu_l^*\omega)^T = P(\omega)J_{2g}h(\mu_l)^TP(\omega)^T.$$

We have

$$(P(\omega)J_{2g}h(\mu_l)^TP(\omega)^T)^T = P(\omega)h(\mu_l)J_{2g}^TP(\omega) = -([\mu_l^*\omega], [\omega]).$$

Let $\tilde{P}(\omega) = (|P_1(\omega)| \quad \dots \quad |P_{2g}(\omega)|)$.

We assume that $\sum_{l=1}^m (r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l))$ is positive definite. Then

$$\begin{aligned} 0 &= \Phi \left(\sum_{l=1}^m r_l [([\omega], [\mu_l^*\omega]) - ([\mu_l^*\omega], [\omega])] \right) \\ &= \Phi \left(P(\omega) \sum_{l=1}^m r_l [J_{2g}h(\mu_l)^T - h(\mu_l)J_{2g}] P(\omega)^T \right) \\ &= \Phi \left(P(\omega) \sum_{l=1}^m r_l [B(\mu_l) + C(\mu_l)] P(\omega)^T \right) \end{aligned}$$

$$\begin{aligned}
&= - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l) |P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l) \langle P_i(\omega), P_j(\omega) \rangle \right) \\
&\leq - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l) |P_i(\omega)|^2 - \sum_{i,j=1}^{2g} |r_l| |c_{ij}(\mu_l)| |P_i(\omega)| |P_j(\omega)| \right) \\
&= -\tilde{P}(\omega) \left(\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right) \right) \tilde{P}(\omega)^T \leq 0.
\end{aligned}$$

Hence $\tilde{P}(\omega) = 0$.

We assume that $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ is negative definite. Then

$$\begin{aligned}
0 &= \Phi \left(\sum_{l=1}^m r_l [(\llbracket \omega \rrbracket, \llbracket \mu_l^* \omega \rrbracket) - (\llbracket \mu_l^* \omega \rrbracket, \llbracket \omega \rrbracket)] \right) \\
&= - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l) |P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l) \langle P_i(\omega), P_j(\omega) \rangle \right) \\
&\geq - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l) |P_i(\omega)|^2 + \sum_{i,j=1}^{2g} |r_l| |c_{ij}(\mu_l)| |P_i(\omega)| |P_j(\omega)| \right) \\
&= -\tilde{P}(\omega) \left(\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right) \right) \tilde{P}(\omega)^T \geq 0.
\end{aligned}$$

Hence $\tilde{P}(\omega) = 0$.

A one-form ω is exact if and only if $\tilde{P}(\omega) = 0$. Hence, Theorem 1.1 holds. \square

3. Quaternionic-valued one-forms

We have a similar condition for quaternionic-valued one-forms.

Proof of Theorem 1.2. We have

$$\begin{aligned}
(\llbracket \omega \rrbracket, \llbracket \mu_l^* \bar{\omega} \rrbracket) &= P(\omega) J_{2g} P(\mu_l^* \bar{\omega})^T = P(\omega) J_{2g} h(\mu_l)^T P(\bar{\omega})^T, \\
\overline{(\llbracket \omega \rrbracket, \llbracket \mu_l^* \bar{\omega} \rrbracket)} &= \overline{(P(\omega) J_{2g} P(\mu_l^* \bar{\omega})^T)^T} \\
&= -P(\omega) h(\mu_l) J_{2g} P(\bar{\omega})^T = -(\llbracket \mu_l^* \omega \rrbracket, \llbracket \bar{\omega} \rrbracket).
\end{aligned}$$

Then

$$\begin{aligned}
2 \operatorname{Re}(\llbracket \omega \rrbracket, \llbracket \mu_l^* \bar{\omega} \rrbracket) &= (\llbracket \omega \rrbracket, \llbracket \mu_l^* \bar{\omega} \rrbracket) + \overline{(\llbracket \omega \rrbracket, \llbracket \mu_l^* \bar{\omega} \rrbracket)} \\
&= P(\omega) J_{2g} h(\mu_l)^T P(\bar{\omega})^T - P(\omega) h(\mu_l) J_{2g} P(\bar{\omega})^T \\
&= P(\omega) (J_{2g} h(\mu_l)^T - h(\mu_l) J_{2g}) P(\bar{\omega})^T \\
&= P(\omega) (B(\mu_l) + C(\mu_l)) P(\bar{\omega})^T.
\end{aligned}$$

We assume that $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ is positive definite. Since $\langle u, v \rangle = (u\bar{v} + \bar{v}u)/2$ for $u, v \in \mathbb{H}$, we have

$$\begin{aligned}
0 &= \sum_{l=1}^m r_l [(\llbracket \omega \rrbracket, \llbracket \mu_l^* \bar{\omega} \rrbracket) - (\llbracket \mu_l^* \omega \rrbracket, \llbracket \bar{\omega} \rrbracket)] \\
&= P(\omega) \sum_{l=1}^m r_l [B(\mu_l) + C(\mu_l)] P(\bar{\omega})^T \\
&= - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l) |P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l) \langle P_i(\omega), P_j(\omega) \rangle \right) \\
&\leq - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l) |P_i(\omega)|^2 - \sum_{i,j=1}^{2g} |r_l| |c_{ij}(\mu_l)| |P_i(\omega)| |P_j(\omega)| \right) \\
&= -\tilde{P}(\omega) \left(\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right) \right) \tilde{P}(\bar{\omega})^T \leq 0.
\end{aligned}$$

Hence $\tilde{P}(\omega) = 0$.

We assume that $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ is negative definite. Then

$$\begin{aligned}
0 &= \sum_{l=1}^m r_l [(\llbracket \omega \rrbracket, \llbracket \mu_l^* \bar{\omega} \rrbracket) - (\llbracket \mu_l^* \omega \rrbracket, \llbracket \bar{\omega} \rrbracket)] \\
&= - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ij}(\mu_l) |P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l) \langle P_i(\omega), P_j(\omega) \rangle \right) \\
&\geq - \sum_{l=1}^m \left(\sum_{i=1}^{2g} r_l b_{ij}(\mu_l) |P_i(\omega)|^2 + \sum_{i,j=1}^{2g} |r_l| |c_{ij}(\mu_l)| |P_i(\omega)| |P_j(\omega)| \right) \\
&= -\tilde{P}(\omega) \left(\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right) \right) \tilde{P}(\bar{\omega})^T \geq 0.
\end{aligned}$$

Hence $\tilde{P}(\omega) = 0$.

A one-form ω is exact if and only if $\tilde{P}(\omega) = 0$. Hence, Theorem 1.2 holds. \square

Proof of Corollary 1.3. Since η is not exact, we have

$$\int_M r (\eta \wedge \mu^* \bar{\eta} - \mu^* \eta \wedge \bar{\eta}) \neq 0.$$

On the other hand,

$$\int_M r (\eta \wedge \mu^* \bar{\eta} - \mu^* \eta \wedge \bar{\eta}) = -r \left(\int_{a_1} \eta \int_{b_1} \mu^* \eta + \int_{b_1} \mu^* \eta \int_{a_1} \eta \right)$$

$$= -2r \left\langle \int_{a_1} \eta, \int_{b_1} \mu^* \eta \right\rangle$$

by Lemma 2.1. Hence Corollary 1.3 holds. \square

4. One-forms on tori

We review the classification of tori and their holomorphic automorphisms, and consider Theorem 1.1 and Theorem 1.2 in the case where M is a torus.

Let M be a torus. We consider M as a Riemann surface. Then M is biholomorphic to an orbit space $\mathbb{C}/\Lambda_\lambda$ with a lattice

$$\Lambda_\lambda := \mathbb{Z} + \mathbb{Z}\lambda, \quad \text{Im } \lambda > 0, \quad -\frac{1}{2} < \text{Re } \lambda \leq \frac{1}{2}, \quad \begin{cases} |\lambda| \geq 1 & (\text{Re } \lambda \geq 0), \\ |\lambda| > 1 & (\text{Re } \lambda < 0). \end{cases}$$

The torus \mathbb{C}/Λ_i is called a *square torus*. The torus $\mathbb{C}/\Lambda_{e^{\pi i/3}}$ is called a *hexagonal torus*. The projection $\psi_\lambda: \mathbb{C} \rightarrow \mathbb{C}/\Lambda_\lambda$ is the universal covering. We define $\tilde{a}: [0, 1] \rightarrow \mathbb{C}$ and $\tilde{b}: [0, 1] \rightarrow \mathbb{C}$ by $\tilde{a}(t) := t$ and $\tilde{b}(t) := \lambda t$ respectively. Then $a := \psi_\lambda \circ \tilde{a}$ and $b := \psi_\lambda \circ \tilde{b}$ are closed curves in $\mathbb{C}/\Lambda_\lambda$ subject to the relation $aba^{-1}b^{-1} = 1$. The fundamental group $\pi_1(\mathbb{C}/\Lambda_\lambda, \psi_\lambda(0))$ is generated by $[a]$ and $[b]$.

A map $\tau: \mathbb{C}/\Lambda_\lambda \rightarrow \mathbb{C}/\Lambda_\lambda$ is a holomorphic automorphism such that τ^2 is the identity map, if and only if $(\tau \circ \psi_\lambda)(z) = \psi_\lambda(\pm z)$. There exists a holomorphic automorphism τ such that τ^2 is not the identity map, if and only if $\mathbb{C}/\Lambda_\lambda$ is a square torus or a hexagonal torus. In fact, we define $\tilde{\tau}_{m,n}: \mathbb{C} \rightarrow \mathbb{C}$ by

$$\tilde{\tau}_{m,n}(z) = e^{2\pi m i/n} z \quad (n = 4, 6, \quad m = 0, 1, \dots, n-1).$$

Then $\tau_{m,6}: \mathbb{C}/\Lambda_{e^{\pi i/3}} \rightarrow \mathbb{C}/\Lambda_{e^{\pi i/3}}$ is defined by $\tau_{m,6} \circ \psi_{e^{\pi i/3}} := \psi_{e^{\pi i/3}} \circ \tilde{\tau}_{m,6}$. A map $\tau_{m,6}$ is a holomorphic automorphism of a hexagonal torus. Similarly, $\tau_{m,4}: \mathbb{C}/\Lambda_i \rightarrow \mathbb{C}/\Lambda_i$ is defined by $\tau_{m,4} \circ \psi_i := \psi_i \circ \tilde{\tau}_{m,4}$. A map $\tau_{m,4}$ is a holomorphic automorphism of a square torus.

Corollary 4.1. *Let M be a square torus and ω a one-form on M with values in $\mathbb{R}^n \subset C\ell_n$, satisfying $\omega \wedge \tau_{1,4}^* \bar{\omega} - \tau_{1,4}^* \omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$\Phi([\omega], [\tau_{1,4}^* \omega]) - ([\tau_{1,4}^* \omega], [\omega]) = 0.$$

Proof. Let M be a square torus. We see that

$$h(\tau_{1,4}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $m = 1$, $r_1 = 1$, and $\mu_1 = \tau_{1,4}$. Then

$$J_2 h(\tau_{1,4})^T - h(\tau_{1,4}) J_2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = B(\tau_{1,4}),$$

$$C(\tau_{1,4}) = \tilde{C}(\tau_{1,4}) = 0,$$

$$B(\tau_{1,4}) + \tilde{C}(\tau_{1,4}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then Corollary 4.1 holds by Theorem 1.1. \square

Corollary 4.2. *Let M be a hexagonal torus and ω a one-form on M with values in $\mathbb{R}^n \subset C\ell_n$, satisfying $\omega \wedge \tau_{1,6}^* \bar{\omega} - \tau_{1,6}^* \omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if*

$$\Phi([\omega], [\tau_{1,6}^* \omega]) - ([\tau_{1,6}^* \omega], [\omega]) = 0.$$

Proof. Let M be a hexagonal torus. We see that

$$h(\tau_{1,6}) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Let $m = 1$, $r_1 = 1$, and $\mu_1 = \tau_{1,6}$. Then

$$\begin{aligned} J_2 h(\tau_{1,6})^T - h(\tau_{1,6}) J_2 &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \\ B(\tau_{1,6}) &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad C(\tau_{1,6}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{C}(\tau_{1,6}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ B(\tau_{1,6}) + \tilde{C}(\tau_{1,6}) &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \end{aligned}$$

Then Corollary 4.2 holds by Theorem 1.1. \square

We collect statements for quaternionic-valued one-forms, which are obtained in a similar fashion as above. We omit their proof.

Corollary 4.3. *Let M be a square torus and ω a one-form on M with values in $\mathbb{R}^4 \cong \mathbb{H}$, satisfying $\omega \wedge \tau_{1,4}^* \bar{\omega} - \tau_{1,4}^* \omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$([\omega], [\tau_{1,4}^* \bar{\omega}]) - ([\tau_{1,4}^* \omega], [\bar{\omega}]) = 0.$$

Corollary 4.4. *Let M be a hexagonal torus and ω a one-form on M with values in $\mathbb{R}^4 \cong \mathbb{H}$, satisfying $\omega \wedge \tau_{1,6}^* \bar{\omega} - \tau_{1,6}^* \omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if*

$$([\omega], [\tau_{1,6}^* \bar{\omega}]) - ([\tau_{1,6}^* \omega], [\bar{\omega}]) = 0.$$

5. One-forms on a hyperelliptic curve

We review a hyperelliptic curve and its automorphisms, and prove Corollary 5.1.

Let M be the hyperelliptic curve of genus g with affine model (5.1) and τ be a holomorphic automorphism of M defined by (5.2). If $g = 1$, then M is a square torus and $\tau = \tau_{1,4}$.

We label two copies of a sphere, which is identified with $\Sigma = \mathbb{C} \cup \{\infty\}$, sheet I and sheet II. On each sheet, we draw a smooth curve, joining $e^{(2k-1)\pi i/(g+1)}$ and $e^{2k\pi i/(g+1)}$ ($k = 1, \dots, g+1$). These curves are called cuts. We assume that these cuts do not intersect each other. Each cut has

two banks, called the N-bank and the S-bank. The surface M is constructed by joining every S-bank on sheet I to an N-bank of the corresponding cut on sheet II, and joining every N-bank on sheet I to an S-bank of the corresponding cut on sheet II. We draw a simple closed curve a_k , winding counterclockwise once, around the cuts joining $e^{(2k-1)\pi i/(g+1)}$ and $e^{2k\pi i/(g+1)}$ on sheet I ($k = 1, \dots, g$). We choose a curve b_k starting from a point on the cut from $e^{(2g+1)\pi i/(g+1)}$ to 1, going on sheet I, to a point on the cut from $e^{(2k-1)\pi i/(g+1)}$ to $e^{2k\pi i/(g+1)}$, and returning on the sheet II ($k = 1, \dots, g$). A map $\tau: M \rightarrow M$ defined by $\tau(w, z) = (-w, e^{\pi i/(g+1)}z)$ is a holomorphic automorphism of M . We have a situation similar to a square torus, when a Riemann surface is hyperelliptic. Let M be a hyperelliptic curve of genus g with affine plane model

$$\left\{ (z, w) \in \mathbb{C} \times \mathbb{C} \mid w^2 = z^{2(g+1)} - 1 \right\}. \quad (5.1)$$

A holomorphic automorphism $\tau: M \rightarrow M$ is defined by

$$\tau(z, w) := \left(e^{\pi i/(g+1)}z, -w \right). \quad (5.2)$$

We have

$$h(\tau) = \begin{pmatrix} O & Q(\tau) \\ R(\tau) & O \end{pmatrix},$$

$$Q(\tau) = \begin{pmatrix} -1 & \dots\dots\dots & -1 \\ 0 & -1 & \dots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 \end{pmatrix},$$

$$R(\tau) = \begin{pmatrix} 1 & 0 & \dots\dots\dots & 0 \\ -1 & 1 & 0 & \dots\dots\dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 & 0 \\ 0 & \dots\dots\dots & 0 & -1 & 1 \end{pmatrix}.$$

Corollary 5.1. *Let M be a hyperelliptic curve of genus two with affine model (5.1), and τ a holomorphic automorphism defined by (5.2). Let ω be a one-form with values in $\mathbb{R}^n \subset \mathcal{C}\ell_n$, satisfying $\omega \wedge \tau^*\omega - \tau^*\omega \wedge \omega \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$\Phi(([\omega], [\tau^*\omega]) - ([\tau^*\omega], [\omega])) = 0.$$

Proof. We have

$$h(\tau) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

Let $m = 1$, $r_1 = 1$, and $\mu_1 = \tau_{1,6}$. Then

$$\begin{aligned}
 J_4 h(\tau)^T - h(\tau) J_4 &= \begin{pmatrix} -2 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \\
 B(\tau) &= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad C(\tau) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
 \tilde{C}(\tau) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
 B(\tau) + \tilde{C}(\tau) &= \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.
 \end{aligned}$$

Then Corollary 4.2 holds by Theorem 1.1. \square

As in the previous section, we have a similar statement as above for quaternionic-valued one-forms. We omit the proof.

Corollary 5.2. *Let M be a hyperelliptic curve of genus two with affine model (5.1), and τ a holomorphic automorphism defined by (5.2). Let ω be a one-form with values in $\mathbb{R}^4 \cong \mathbb{H}$, satisfying $\omega \wedge \tau^* \bar{\omega} - \tau^* \omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$([\omega], [\tau^* \bar{\omega}]) - ([\tau^* \omega], [\bar{\omega}]) = 0.$$

6. Surfaces in Euclidean four-space

We prove Corollary 1.4 and Corollary 1.5.

Firstly, we review the theory of surfaces in terms of quaternions (see [2]). Let $f: M \rightarrow \mathbb{H}$ be a surface. Then there exists $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$ such that $*df = Ndf$. Let $\mathcal{H}: M \rightarrow \mathbb{H}$ be the mean curvature vector of f . Then the following equation holds.

$$-2df \bar{\mathcal{H}} = *dN + N dN.$$

Differentiating this equation, we have

$$2df \wedge d\bar{\mathcal{H}} = d*dN + dN \wedge dN.$$

Proof of Corollary 1.4. We assume that N , \mathcal{H} , and ω satisfy the assumption. Then

$$2d\omega \bar{\mathcal{H}} = 2d(\omega \bar{\mathcal{H}}) + 2\omega \wedge d\bar{\mathcal{H}} = 0.$$

Hence ω is a closed one-form. By Theorem 1.1, the one-form ω is exact. Then there exists a map $f: M \rightarrow \mathbb{H}$ with $df = \omega$. By the assumption for ω , we have $*df = N df$. Hence, f is a surface with $*df = N df$ and mean curvature vector field \mathcal{H} . \square

The following is the case where a surface takes values in $\mathbb{R}^3 \cong \text{Im } \mathbb{H}$.

Corollary 6.1. *We assume that there exist $\mu_1, \dots, \mu_m \in \mathcal{A}$ and $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ is positive definite,
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ is negative definite.

We assume that maps $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$ and $H: M \rightarrow \mathbb{R}$, and a non-zero one-form ω on M satisfy

$$2\omega H = N * dN - dN, \quad -2\omega \wedge (dH N + H dN) = d * dN + dN \wedge dN,$$

$$\sum_{l=1}^m r_l (\omega \wedge \mu_l^* \bar{\omega} - \mu_l^* \omega \wedge \bar{\omega}) \neq 0,$$

$$\sum_{l=1}^m r_l ([[\omega], [\mu_l^* \bar{\omega}]] - ([[\mu_l^* \omega], [\bar{\omega}]]) = 0.$$

Then there exists a surface $f: M \rightarrow \text{Im } \mathbb{H} \cong \mathbb{R}^3$ with mean curvature H , such that $df = \omega$ and $*df = N df = -df N$.

Proof. For a surface $f: M \rightarrow \text{Im } \mathbb{H}$, we have $*df = N df = -df N$ with Gauss map $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$. The mean curvature vector of f is $\mathcal{H} = HN$ with mean curvature function H . Then this corollary follows from Corollary 1.4. \square

Proof of Corollary 1.5. A harmonic map $N: M \rightarrow S^2 \subset \text{Im } \mathbb{H}$ satisfies the equation

$$d * dN = N dN \wedge * dN.$$

Let $2\omega H = N * dN - dN$ ($H \in \mathbb{R} \setminus \{0\}$). Then

$$\begin{aligned} -2\omega \wedge (dH N + H dN) &= -(N * dN - dN) \wedge dN \\ &= -N * dN \wedge dN + dN \wedge dN = d * dN + dN \wedge dN. \end{aligned}$$

Then Corollary 1.5 follows from Corollary 6.1. \square

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